is satisfied. Relation (2.10) can be obtained using the equations of rotation of the disk under the action of impact forces due to friction

$$J_0\Delta\omega = -MS;\quad J_0 = J_c + mc^2 \tag{2.11}$$

where $\Delta\omega$ is the angular velocity increment during the impact.

The equation of motion of the centre of mass leads in this case to the equation

$$mc\Delta\omega = -\Phi_\xi S \tag{2.12}$$

Under conditions of the motion considered here M and Φ_ξ should be taken in (2.11) and (2.12) in the form /1/.

$$M = 2h_2a_0,\quad \Phi_\xi = h_1a_0$$

From (2.11) and (2.12) we obtain the linear dependence of the angular velocity increment on the normal reaction momentum

$$\Delta\omega = -\frac{4f}{3am}S = -\frac{4f}{3a}\Delta u$$

where in addition to (2.10) the value of the coefficient $a_0 = \pi a^3/3$ is taken onto account.

For practical purposes we will formulate this property in the form of a statement. If a plane rigid body rotates about an axis passing through the point O normal to the body plane, then at the instant of collisional start of frictional braking (along the axis of rotation) with circular contact area, the axis does not experience transverse impact loads when $a^2 = 2lc$, where l is the reduced length of the physical pendulum.

The last equation follows from a comparison of (2.10) with the requirement that the momentum of the resultant friction forces is applied at the centre of impact. If, for example, $c = a/4$ (the line of zero pressure touches the contour of the circular contact area), the diameter of that circle must be equal to the reduced length. In the trivial case when $c = 0$ we have $V \equiv 0$ for any value of a.

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RELAXATION IN DISSIPATIVE MECHANICAL SYSTEMS*

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An asymptotic expression for long times is obtained for a $2n$-parametric family of solutions of a Hamiltonian system with n degrees of freedom, modified by the addition of generalized dissipative forces. The method used here is based on a preliminary study of the solutions of a linearized system of equations, followed by the application of the Schauder principle in Banach space with a suitably chosen norm.

1. The aim of this paper is to study relaxation in a mechanical system, the equations of motion of which are written in the form

$$p^{\cdot} = -\frac{\partial H(p,q)}{\partial q} + Q(t,p,q),\quad q^{\cdot} = \frac{\partial H(p,q)}{\partial p} \tag{1.1}$$
$$H(p,q) = 2^{-1}(p^2 + q^2) + \Pi_1(q)$$

Here p, q are $(n \times 1)$-vectors (columns) of generalized momenta and coordinates, and the $(n \times 1)$-vector $Q(t,p,q)$ defines the Lagrangian forces in $t \in R_+, p, q$ variables. The expansion $\Pi_1(q)$ in q_i coordinates begins with terms of at least the third order.

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We will further assume that the following condition of total dissipation /1/ holds for the Lagrangian forces:

$$\sum_i p_i Q_i(t, p, q) \leqslant -a(\|p\|) \tag{1.2}$$

where $a(x)$ is a continuous and strictly increasing function, defined on R_+ and such, that $a(0) = 0$. It has been shown that if the condition of total dissipation holds for the system, the potential energy $\Pi(q)$ has an isolated minimum at $q = 0$ and $Q(t, p, q) \to 0$ as $p \to 0$ uniformly in $t \in R_+$, then the origin of the phase space $q = p = 0$ is uniformly asymptotically stable. We note that (1.2) and that fact that Q is continuous, together imply that /1/

$$Q(t, 0, q) = 0 \tag{1.3}$$

We shall show in Sect. 2 that by imposing certain additional restrictions we can find, for every real $(2n \times 1)$-vector c a corresponding $t_1 = t_1(c)$ such that when $t \geqslant t_1$, a real solution of (1.1) exists which has the asymptotic representation

$$y = \left\|\begin{matrix} q \\ p \end{matrix}\right\| = Y(t)(c + o(1)), \quad t \to \infty$$

where $Y(t)$ is a real fundamental solution matrix *(FSM)* of the linear system obtained by linearizing (1.1) about the equilibrium $q = p = 0$. We use here the result obtained in /2,3/ which dealt with the asymptotic solutions of linear differential vector equations of general form, with variable coefficients, to compute the asymptotics of the *FSM* of the linearized system. The asymptotic form in question yields the required estimates for the matrices $Y(t)$, $Y^{-1}(t)$. After this the asymptotic form of the solutions of (1.1) is found using the Schauder principle. Equation (1.1) is replaced by a non-linear integral equation and the latter is studied in the Banach space of vector functions with a specially chosen norm (the choice of the norm depends on the asymptotics of the solutions of the linearized system).

The method of investigating the relaxation in mechanical systems with dissipation developed here can be used in other problems with an asymptotically stable equilibrium position, not necessarily possessing the Hamiltonian form of Eqs.(1.1) modified by the addition of generalized forces. Such problems are encountered not only in mechanics, but also in physics, biology and chemistry.

The constraints imposed here are discussed in Sect.3, and specific examples illustrating the results obtained are given in Sect.4. We note that in fact we use not the condition of total dissipation (1.2), but only the condition of total dissipation of the linear part of the Lagrangian forces. This condition distinguishes the class of mechanical systems for which the results obtained in this paper (assuming, of course, that the remaining constraints also hold) are applicable.

2. Linearizing (1.1) about the equilibrium position $q = p = 0$, we obtain the set of equations

$$q_i^{\cdot} = p_i, \; p_i^{\cdot} = -q_i + \sum_j a_{ij}(t) p_j, \quad i = 1, 2, \ldots, n \tag{2.1}$$

The absence of the terms $\sum_j b_{ij}(t) q_j$ from the second group of Eqs.(2.1) follows from (1.1). We assume that the $(n \times n)$-matrix $A_1(t) = (a_{ij}(t)), a_{ij}(t) = \partial Q_i(t, 0, 0)/\partial p_j$ is symmetric and negative definite (see (1.2)). We will formulate this, and certain other restrictions imposed on A_1, in the form of the following condition.

1°. $A_1(t)$ is a symmetric, negative definite $\forall t \in R_+$-matrix with a finite, also negative definite $\lim_{t\to\infty} A_1(t) = B_1$. The characteristic roots σ_i $(i = 1, 2, \ldots, n)$ of the matrix B_1 are simple (pairwise different) and $\sigma_i \neq -2$. The negative definiteness of B_1 implies that $\sigma_i < 0$. Denoting by $\sigma_i(t)$ the characteristic roots of the matrix $A_1(t)$, we obtain $\lim_{t\to\infty}\sigma_i(t) = \sigma_i$, $\sigma_i(t) < 0$. To compute the asymptotics of the *FSM* of the linear system (2.1) we must have available a definite version of the conditions of regular behaviour of $A_1(t)$ as $t \to \infty$. Following /2/ we will require that the condition

2°. $\qquad A_1(t) \in C_2(0, \infty), \quad \|A_1^{\cdot}\|^2 + \|A_1^{\cdot\cdot}\| \in L_1(0, \infty)$

is satisfied. Equations (2.1) can be written in the form

$$y^{\cdot} = A(t)\, y \tag{2.2}$$

$$y = \left\|\begin{matrix} q(t) \\ p(t) \end{matrix}\right\| - (2n \times 1)\text{- vector},$$

$$A(t) = \left\|\begin{matrix} 0 & E \\ -E & A_1(t) \end{matrix}\right\| - (2n \times 2n)\text{-matrix}$$

Suppose the orthogonal matrix $T_1(t)$ diagonalizes the symmetric matrix $A_1(t)$, and $T_1 = \lim_{t\to\infty} T_1(t)$. Then

$$T_2^{-1}(t)\,A(t)\,T_2(t)=\begin{Vmatrix}0 & E\\ -E & S(t)\end{Vmatrix}=R(t)$$

$$T_2^{-1}BT_2=\begin{Vmatrix}0 & E\\ -E & S\end{Vmatrix}=R,\quad R=\lim_{t\to\infty}R(t),\quad B=\lim_{t\to\infty}A(t)$$

$$S(t)=\operatorname{diag}\{\sigma_1(t),\ldots,\sigma_n(t)\},\quad S=\lim_{t\to\infty}S(t)=\operatorname{diag}\{\sigma_1,\ldots,\sigma_n\}$$

$$T_2(t)=\begin{Vmatrix}T_1(t) & 0\\ 0 & T_1(t)\end{Vmatrix},\quad T_2=\lim_{t\to\infty}T_2(t)=\begin{Vmatrix}T_1 & 0\\ 0 & T_1\end{Vmatrix}$$

The characteristic polynomial of the matrix $R(t)$ is found using the Schur formula /4/, and is equal to

$$|R(t)-\mu E|=\prod_{i=1}^{n}(\mu^2-\sigma_i(t)\mu+1)$$

Consequently, the characteristic roots $\lambda_{i,\,n+i}$ of the matrix B and $\lambda_{i,\,n+i}(t)$ of the matrix $A(t)$ have the form (every $i=1,2,\ldots,n$ has the corresponding two roots $\lambda_{i,\,n+i}$)

$$\lambda_{i,\,n+i}=\frac{\sigma_i}{2}\pm\left(\frac{\sigma_i^2}{4}-1\right)^{1/2} \tag{2.3}$$

$$\lambda_{i,\,n+i}(t)=\frac{\sigma_i(t)}{2}\pm\left(\frac{\sigma_i^2(t)}{4}-1\right)^{1/2} \tag{2.4}$$

From (2.3), (2.4) and condition 1° (the simple form of $\sigma_i\neq-2$) implies that the roots $\lambda_{i,\,n+i}$ of the matrix B and $\lambda_{i,\,n+i}(t)$ of the matrix $A(t)$ are simple for $t>t_0>1$, and the roots σ_i can be numbered in such a manner that for $\nu_j(t)=\operatorname{Re}\lambda_j(t)$ when $t\geqslant t_0$ we have $\max_j\nu_j(t)=\nu_1(t)$, $\min_j\nu_j(t)=\nu_{2n}(t)$, $j=1,2,\ldots,2n$. Then $0>\nu_1(t)\geqslant\ldots\geqslant\nu_{2n}(t)$. The equal sign can occur here only between the real parts of the complex conjugate roots. We introduce the following condition for the non-linear part $Q_H=Q-A_1(t)\,p$ of the Lagrangian forces and the $(n\times1)$-vector $\partial\Pi_1/\partial q$.

3°. The following estimate holds in the region $\Omega_\delta=\{y:\|y\|=\sum_i(|q_i|+|p_i|)<\delta<1\}$ of the phase space (q,p) uniformly in $t\in R_+$:

$$\left\|Q_H-\frac{\partial\Pi_1}{\partial q}\right\|\leqslant c_1\|y\|^m \tag{2.5}$$

where $c_1>0$ is a constant and $m>1$ is such, that $d=m\nu_1-\nu_{2n}<0$; here $\nu_i=\lim_{t\to\infty}\nu_i(t)$. We can now formulate the following fundamental result.

Theorem 1. Let condition (1.2) and conditions 1°–3° all hold. Then for every real $(2n\times1)$-vector c we have $t_1=t_1(c)\gg1$ such that when $\forall t\geqslant t_1$, Eq.(1.1) has a real solution $y(t)$ for which the following asymptotic representation holds:

$$y(t)=\begin{Vmatrix}q(t)\\ p(t)\end{Vmatrix}=Y(t)\,(c+o(1)) \tag{2.6}$$

where $Y(t)$ is the real *FSM* of (2.2).

The following representation holds for $Y(t)$:

$$Y(t)=T\,(E+o(1))\exp\int_{t_0}^{t}\Lambda(t)\,dt\,C \tag{2.7}$$

$$\Lambda(t)=\operatorname{diag}\{\lambda_1(t),\ldots,\lambda_{2n}(t)\}$$

where T is a constant non-degenerate matrix which converts the matrix B to its diagonal form, and C is also a constant non-degenerate matrix.

Proof. We will first establish the validity of (2.7) for the *FSM* $Y(t)$, and then the validity of the asymptotic forms (2.6) for the solution of (1.1). Since the roots $\lambda_{i,\,n+i}(t)$ are simple for $t\geqslant t_0$, a matrix $T_3(t)$ exists for these roots, which diagonalizes the matrix $R(t)$. Then $T_3=\lim_{t\to\infty}T_3(t)$ diagonalizes R and $T(t)=T_2(t)\,T_3(t)-A(t)$. We obtain the asymptotic forms of $Y(t)$ with the help of the matrices $T(t)$, $T^{-1}(t)$ and the diagonal matrix

$$\Lambda_1(t)=\operatorname{diag}\{\lambda_{11}(t),\ldots,\lambda_{12n}(t)\}=-\operatorname{diag}\{T^{-1}(t)\,T^{\cdot}(t)\}$$

Thus we have $\lambda_{1i}=-(T^{-1}(t)\,T^{\cdot}(t))_{ii}$ (the smoothness of $T(t)$ is identical with the smoothness of $A(t)$). The matrices $T_3(t)$, $T_3^{-1}(t)$ satisfy the matrix equations

$$R(t)\,T_3(t)=T_3(t)\,\Lambda(t) \tag{2.8}$$

$$T_3^{-1}(t)\,R(t)=\Lambda(t)\,T_3^{-1}(t) \tag{2.9}$$

$$T_3^{-1}(t)\,T_3(t)=E \tag{2.10}$$

From (2.8) and condition 1° it follows (below we omit the argument t from the elements $t_{ij}(t)$ of the matrix $T_3(t)$, and $\tau_{ij}(t)$ of the matrix $T_3^{-1}(t)$, as well as from the roots $\lambda_i(t)$, $\sigma_i(t)$), that $t_{ij}=t_{n+ij}=t_{in+j}=t_{n+in+j}=0$ for $i\neq j$, $i,j=1,2,\ldots,n$, $t_{n+jj}=\lambda_j t_{jj}$, $t_{n+jn+j}=\lambda_{n+j}t_{jn+j}$, and t_{jj}, t_{jn+j} remain arbitrary.

Putting $t_{jj}=t_{jn+j}=1$, we obtain

$$T_3(t)=\begin{Vmatrix}E & E\\ \Lambda_2 & \Lambda_3\end{Vmatrix}$$

$$\Lambda_2=\operatorname{diag}\{\lambda_1,\ldots,\lambda_n\},\quad \Lambda_3=\operatorname{diag}\{\lambda_{n+1},\ldots,\lambda_{2n}\}$$

From (2.9) it follows that $\tau_{ij}=\tau_{in+j}=0$ for $i\neq j$, $n+j$, $\tau_{jn+j}=-\lambda_j\tau_{jj}$, $\tau_{n+jn+j}=-\lambda_{n+j}\tau_{n+jj}$ and τ_{jj}, τ_{n+jj}, $j=1,2,\ldots,n$ in (2.9) remain undetermined. To determine them we compute the diagonal elements in (2.10). We obtain

$$\tau_{jj}t_{jj}+\tau_{jn+j}t_{n+jj}=1,\quad \tau_{n+jj}t_{jn+j}+\tau_{n+jn+j}t_{n+jn+j}=1$$

Whence, substituting the already determined t_{jj}, t_{n+jj}, t_{jn+j}, t_{n+jn+j}, τ_{jn+j}, τ_{n+jn+j}, we obtain τ_{jj}, τ_{n+jj}. Since $\lambda_j\lambda_{n+j}=1$, it follows that the final formulas for the elements of the matrix $T_3^{-1}(t)$, different from zero, have the form

$$\tau_{jn+j}=(\lambda_j-\lambda_{n+j})^{-1},\quad \tau_{n+jn+j}=(\lambda_{n+j}-\lambda_j)^{-1}$$
$$\tau_{jj}=\lambda_{n+j}/(\lambda_{n+j}-\lambda_j),\quad \tau_{n+jj}=\lambda_j/(\lambda_j-\lambda_{n+j})$$

Now we can compute the matrix Λ_1. Clearly (here again we omit the argument t)

$$T^{-1}T^{\cdot}=T_3^{-1}(T_2^{-1}T_2^{\cdot})\,T_3+T_3^{-1}T_3^{\cdot} \tag{2.11}$$

The matrix $P=T_2^{-1}T_2^{\cdot}$ is a diagonal block matrix and its diagonal blocks $T_1^{-1}T_1^{\cdot}$ are skew symmetric by virtue of the orthogonality of T_1. Computing the i-th diagonal element in the first term on the right-hand side of (2.11) and taking into account the structure of the matrices T_3, T_3^{-1}, we find that for $i=1,2,\ldots,n$ it is equal to

$$\sum_{k,r=1}^{n}\tau_{ik}p_{kr}t_{ri}=\tau_{ii}(p_{ii}t_{ii}+p_{in+i}t_{n+ii})+\tau_{in+i}(p_{n+ii}t_{ii}+p_{n+in+i}t_{n+ii})=0$$

since by virtue of the skew symmetric character of P, $p_{ii}=p_{n+in+i}=0$, while $p_{in+i}=p_{n+ii}=0$, since P is a diagonal block matrix. A similar result is obtained for $i=n+1,n+2,\ldots,2n$, therefore $(T^{-1}T^{\cdot})_{ii}=(T_3^{-1}T_3^{\cdot})_{ii}$, so that

$$\lambda_{1i}=\lambda_i^{\cdot}(\lambda_{n+i}-\lambda_i)^{-1},\quad \lambda_{1n+i}=\lambda_{n+i}^{\cdot}(\lambda_i-\lambda_{n+i})^{-1}$$

Equations (2.4) can now be used to express the elements of the matrix Λ_1 directly in terms of the roots $\sigma_i(t)$ of the matrix $A_1(t)$

$$\lambda_{1i}=-\frac{\sigma_i^{\cdot}}{2}\left(\frac{\sigma_i(t)}{\sigma_i^2(t)-4}+(\sigma_i^2(t)-4)^{-1/2}\right),\quad i=1,2,\ldots,n,\quad \lambda_{1n+i}=-\frac{\sigma_i^{\cdot}}{2}\left(\frac{\sigma_i(t)}{\sigma_i^2(t)-4}-(\sigma_i^2(t)-4)^{-1/2}\right) \tag{2.12}$$

From condition 2° it follows that $\|A_1^{\cdot}\|=o(1)$ and hence $\sigma_i^{\cdot}=o(1)$, and from (2.12) we have

$$\lambda_{1i}=o(1),\quad i=1,2,\ldots,2n \tag{2.13}$$

Let us now consider the function

$$I_{ij}(t)=\operatorname{Re}(\lambda_i(t)+\lambda_{1i}(t)-\lambda_j(t)-\lambda_{1j}(t)),\quad t\geqslant t_0\gg 1,\qquad i,j=1,2,\ldots,2n;\ i\neq j$$

When $j\neq i+n$ we obtain from condition 1° and (2.13), the estimate

$$|I_{ij}(t)|\geqslant\beta>0 \tag{2.14}$$

where β is a constant. The estimate (2.14) also holds when $j=i+n$, $\lim_{t\to\infty}\sigma_i(t)=\sigma_i<-2$. This again follows from 1°, (2.4) and (2.13) (provided that $\sigma_i<-2$, $\lambda_{i,n+i}(t)$, $\lambda_{1i}(t)$, $\lambda_{1n+i}(t)$ are real for $t\geqslant t_0$). If on the other hand $0>\sigma_i>-2$, then from (2.4) and (2.12) we find that $\lambda_{n+i}(t)=\bar\lambda_i(t)$, $\lambda_{1n+i}(t)=\bar\lambda_{1i}(t)$, so that

$$I_{in+i}(t)\equiv 0,\quad t\geqslant t_0 \tag{2.15}$$

The estimate (2.14) and identity (2.15) show that conditions 1°-2° enable Theorem 3.1 of /2/ to be used when computing the asymptotic forms of the *FSM* of (2.2). Using this theorem we find that when $t\geqslant t_0$, (2.2) has the following *FSM:*

$$Y_1(t)=T(t)(E+o(1))\exp\int_{t_0}^{t}(\Lambda+\Lambda_1)\,dt \tag{2.16}$$

But (2.12) and condition 1° imply that the following finite limit exists:

$$\lim_{t\to\infty}\int_{t_0}^{t}\Lambda_1(t)\,dt=C_1$$

Multiplying the matrix $Y_1(t)$ on the right by the constant matrix $\exp(-C_1)$ we find, that when conditions 1° and 2° hold, Eq.(2.2) has the *FSM* $Y_2(t)$ for which the following representation holds:

$$Y_2(t) = T(E + o(1)) \exp \int_{t_0}^{t} \Lambda \, dt \tag{2.17}$$

($T = \lim_{t\to\infty} T(t)$ is a constant non-degenerate matrix which converts the matrix B to diagonal form). Clearly, for $Y_2^{-1}(t)$ we have

$$Y_2^{-1}(t) = \exp\left(-\int_{t_0}^{t} \Lambda \, dt\right)(E + o(1))\, T^{-1} \tag{2.18}$$

The matrix Y_2 may be complex (it will be if its roots $\lambda_{i,n+i}(t)$ include the complex roots i.e. if $0 > \sigma_i > -2$ for some i, otherwise Y_2 is real). But the coefficients of (2.2) are real, and therefore the equation has real *FSM*; e.g. such will be the *FSM* of $Y(t)$ satisfying the initial condition $Y(0) = E$. Since any two *FSM* of (2.2) differ from each other only by a non-degenerate constant right multiplier, it follows that the real *FSM* has the form $Y(t) = Y_2(t)\, C$, where C is a non-degenerate matrix (generally speaking the matrix is complex, however, when all $\sigma_i < -2$, we can put $C = E$). The representation (2.7) for $Y(t)$ follows from (2.17). From (2.17), (2.18) and the condition of sequential numbering of the roots σ_i, we obtain the following estimates which will play an important role below ($c_{2,3} > 0$ are certain constants)

$$\| Y(t) \| \leqslant c_2 \exp \int_{t_0}^{t} \nu_1(\tau)\, d\tau, \quad t \geqslant t_0 \tag{2.19}$$

$$\| Y^{-1}(s) \| \leqslant c_3 \exp \int_{s}^{t_0} \nu_{2n}(\tau)\, d\tau, \quad s \geqslant t_0 \tag{2.20}$$

Let us now compute the asymptotic forms of the $2n$-parameter family of solution of (1.1), writing it first in the form (f is a $(2n \times 1)$-vector)

$$y^{\cdot} = A(t)\, y + f \tag{2.21}$$

$$y = \begin{Vmatrix} q(t) \\ p(t) \end{Vmatrix}, \quad A(t) = \begin{Vmatrix} 0 & E \\ -E & A_1(t) \end{Vmatrix}, \quad f = \begin{Vmatrix} 0 \\ Q_{\text{H}} - \dfrac{\partial \Pi_1}{\partial q} \end{Vmatrix}$$

We change from (2.21) to the integral equation (c is an arbitrary real $(2n \times 1)$-vector)

$$y(t) = Y(t)\, c - \int_{t}^{\infty} Y(t)\, Y^{-1}(s)\, f(s, y(s))\, ds = (Iy)(t) \tag{2.22}$$

Clearly, t_0 can be chosen so that (see condition 3°)

$$m\nu_1(t) - \nu_{2n}(t) \leqslant d/2, \quad \nu_1(t) \leqslant \nu_1/2, \quad t \geqslant t_0 \tag{2.23}$$

Suppose now that α is a constant satisfying the inequality

$$0 < \alpha < \min\{-d/(4m),\ -\nu_1/4\}$$

We introduce the Banach space W of $(2n \times 1)$-vector functions

$$y(t) = \begin{Vmatrix} q(t) \\ p(t) \end{Vmatrix}$$

continuous for $t \geqslant t_1 \geqslant t_0$ (t_1 will be chosen later) with the norm

$$\| y(t) \|_1 = \sup_{t \geqslant t_1} \{\varphi(t) \sum_i (|q_i(t)| + |p_i(t)|)\} < \infty$$

$$\left(\varphi(t) = \exp\left(-\alpha(t - t_0) - \int_{t_0}^{t} \nu_1(\tau)\, d\tau\right)\right)$$

and consider the sphere $W_\delta = \{y \in W\colon \| y \|_1 \leqslant \delta\}$, where $\delta > 0$ is the same as in condition 3°.

First, we shall show that the operator I given by (2.22) leaves the sphere W_δ invariant, provided $t_1 = t_1(\delta, c)$ is sufficiently large. By virtue of the definition of the norm in W, the inclusion $y \in W_\delta$ implies that $\forall t \geqslant t_1$, the point $(q(t), p(t))$ of the phase space belongs to Ω_δ, and the estimate (2.5), which can now be written in the form

$$\| f(s, y(s)) \| \leqslant c_1 \delta \exp\left(m\alpha(s - t_0) + m \int_{t_0}^{s} \nu_1(\tau)\, d\tau\right), \quad y \in W_\delta \tag{2.24}$$

(since $\delta < 1$, therefore $\delta^m < 1$), holds for $y \in W_\delta$

To simplify the notation used in subsequent derivations, we will write

$$\psi(t) = \int_{t}^{\infty} \exp\left\{m\alpha(s - t_0) + \int_{t_0}^{s} (m\nu_1(\tau) - \nu_{2n}(\tau))\, d\tau\right\} ds$$

The convergence of the integral is ensured by the inequality (2.23) and condition of the choice of α. Using the inequalities (2.19), (2.20), (2.24), we obtain

$$\| Iy \|_1 \leqslant \exp(-\alpha(t_1 - t_0))(c_2 \| c \| + \delta c_1 c_2 c_3 \psi(t_1)) \leqslant \delta,$$
$$y \in W_\delta$$

provided that $t_1 = t_1(\delta, c)$ is sufficiently large. The operator I represents the continuous mapping $W_\delta \to W_\delta$.

Let $y_l \to y$ with respect to the norm W, $y_l, y \in W_\delta$. Then, using (2.19) and (2.20) we find, from (2.22)

$$\| Iy_l - Iy \|_1 \leqslant c_2 c_3 \sup_{t \geqslant t_1} \left\{ \exp(-\alpha(t - t_0)) \times \int_t^\infty \exp\left(-\int_{t_0}^s \nu_{2n}(\tau)\, d\tau\right) \| f(s, y_l(s)) - f(s, y(s)) \| ds \right\} \leqslant \tag{2.25}$$
$$c_2 c_3 \int_{t_1}^\infty \exp\left(-\int_{t_0}^s \nu_{2n}(\tau)\, d\tau\right) \| f(s, y_l(s)) - f(s, y(s)) \| ds$$

We must check that $\forall \varepsilon > 0\ \exists \delta_1(\varepsilon) > 0$, is such, that $\| Iy_l - Iy \|_1 \leqslant \varepsilon$, if $\| y_l - y \|_1 \leqslant \delta_1(\varepsilon)$, i.e. if

$$\sum_i (| q_{il}(s) - q_i(s) | + | p_{il}(s) - p_i(s) |) \leqslant \delta_1(\varepsilon) \exp\left(\alpha(s - t_0) + \int_{t_0}^s \nu_1(\tau)\, d\tau\right) \leqslant \delta_1(\varepsilon)$$

But the inclusion $y_l, y \in W_\delta$ yields, when $t \geqslant t_1$, using (2.24),

$$\int_t^\infty \exp\left(-\int_{t_0}^s \nu_{2n}(\tau)\, d\tau\right) \| f(s, y_l(s)) - f(s, y(s)) \| ds \leqslant 2\delta c_1 \psi(t) \tag{2.26}$$

Let us write the integral on the right-hand side of (2.25) as a sum of two integrals

$$\int_{t_1}^\infty = \int_{t_1}^{t_2} + \int_{t_2}^\infty \tag{2.27}$$

From the estimate (2.26) it follows that $t_2 = t_2(\varepsilon)$ can be chosen sufficiently large to ensure that the second term on the right-hand side of (2.27) is not greater that $\varepsilon/(2c_2c_3)$. Next, $\delta_1(\varepsilon)$ can be chosen sufficiently small for it not to exceed $\varepsilon/(2c_2c_3)$ and the first term on the right-hand side of (2.27), by virtue of the uniform continuity of the vector function f. Now from (2.25) the continuity of the operator I in W_δ follows.

To show that the mapping IW_δ is compact, it is sufficient to confirm that for any $\varepsilon > 0$ there is a finite covering of the semiaxis $[t_1, \infty)$ by the open sets on each of which the oscillation of all vector functions $y_1(t) = \varphi(t)(Iy)(t)$ does not exceed $\varepsilon \forall y \in W_\delta$. From (2.22) we obtain, for $t' \geqslant t'' \geqslant t_1$,

$$y_1(t') - y_1(t'') = (\varphi(t') Y(t') - \varphi(t'') Y(t''))c + \varphi(t'') Y(t'') \int_{t''}^{t'} Y^{-1}(s) f(s, y(s))\, ds +$$
$$(\varphi(t'') Y(t'') - \varphi(t') Y(t')) \int_{t'}^\infty Y^{-1}(s) f(s, y(s))\, ds$$

and this, taking the inequalities (2.19), (2.20) and (2.24) into account, yields

$$\| y_1(t') - y_1(t'') \| \leqslant \| \varphi(t') Y(t') - \varphi(t'') Y(t'') \| \times (\| c \| + \delta c_1 c_3 \psi(t_1)) + \delta c_1 c_2 c_3 \exp(-\alpha(t'' - t_0))(\psi(t'') - \psi(t')) \tag{2.28}$$

In addition we have

$$\| \varphi(t') Y(t') - \varphi(t'') Y(t'') \| \leqslant c_2 e^{\alpha t_0} (e^{-\alpha t'} + e^{-\alpha t''}) \tag{2.29}$$

From (2.28) and (2.29) it follows that $\forall \varepsilon > 0\ \exists t_3(\varepsilon)$ is such that

$$\| y_1(t') - y_1(t'') \| \leqslant \varepsilon,\ \forall y \in W_\delta,\ \forall t', t'' \geqslant t_3(\varepsilon)$$

Further, from (2.28) it also follows that the interval $(t_1, t_3(\varepsilon))$ can be covered by a finite number of intervals of length $\delta_2(\varepsilon)$, in each of which the oscillation of the vector functions $y_1(t)$ does not exceed $\varepsilon \forall y \in W_\delta$ (this follows from the uniform continuity of the segment $[t_1, t_3(\varepsilon)]$ of the matrix $\varphi(t) Y(t)$ and the function $\psi(t)$). We have therefore obtained the required finite cover of the semiaxis $[t_1, \infty)$ by open sets in every one of which the oscillation of the vector functions $y_1(t)\ \forall y \in W_\delta$ does not exceed ε.

We can now assert, in accordance with the Schauder principle, that Eq. (2.2) has a solution in W_δ, i.e. a solution $y(t)$ for which the inequality (2.24) holds. Estimating the integral on the right-hand side of (2.22) using (2.24), we obtain

$$\left\| \int_t^\infty Y^{-1}(s) f(s, y(s))\, ds \right\| \leqslant \delta c_1 c_3 \psi(t) = o(1)$$

Therefore (2.22) implies the validity of the asymptotic representation (2.6) for the solution $y(t)$ constructed, depending on $2n$ arbitrary constant (coordinates of the vector c), and this completes the proof of Theorem 1.

Note 1. The proof of Theorem 1 is based on the use of condition 3° and the estimates (2.19) and (2.20) for the real *FSM* of the linearized system. The estimates are obtained from the asymptotic representations (2.17) and (2.18) for this matrix, and their validity is ensured by conditions 1° and 2°. The representations (2.17) and (2.18) and hence the estimates (2.19) and (2.20), will, however, remain valid when condition 2° is replaced by a more general (but also more cumbersome) condition

$2^{\circ\prime}$. For certain integral $k \geqslant 2$,

$$A_1(t) \in C_k(0, \infty), \ \|A_1^{(s)}\| = o(1), \ s = 1, 2, \ldots, k-2;$$
$$\|A_1^{(s)}\|^2 + \|A_1^{(k)}\| \in L_1(0, \infty), \ s = 1, 2, \ldots, k-1$$

The proof of the above assertion uses Theorem 1 of /3/ instead of Theorem 3.1 of /2/. Thus Theorem 1 remains valid when condition 2° in its formulation is replaced by $2^{\circ\prime}$.

Note 2. If system (1.1) is autonomous, i.e. if the matrix $A(t)$ is constant $A(t) \equiv A$, we can write $Y(t) = \exp(tA)$ in (2.6).

3. Let us discuss briefly the conditions 1°–3° of Theorem 1. First we note that if the symmetry of the matrix $A_1(t)$ is not required, the condition of total dissipation leads to the requirement that the symmetric matrix $A_1(t) + A_1^T(t)$, should be negative definite, provided that $A_1(t) \not\equiv 0$ (here the superscript T denotes transposition). The method described in Sect. 2 can also be used in this case, but the condition that the matrix $A_1(t)$ should be symmetric considerably simplifies the derivation as well as the formulation of the final results. In the case when the Lagrangian forces Q do not depend explicitly on t (i.e. system (1.1) is autonomous), the symmetry (constancy) of the matrix A_1 follows from physical considerations, namely from the principle of symmetry of the kinetic Onsager coefficients /5/. Condition 2° is satisfied automatically for an autonomous system. We stress once again the importance of the demand in condition 1° that the matrix $A_1(t)$ be negative definite, following from the condition of total dissipation of the linear part of the Lagrangian forces. It is precisely this condition that defines the class of mechanical systems whose relaxation can be studied using the method developed in this paper. Condition 2° appears as the condition of regularity of behaviour of the matrix $A_1(t)$ elements as $t \to \infty$, and condition 3° represents the condition of "adjacency" of the non-linear parts acting on the system of Lagrangian and potential forces.

Let us now consider a system with one degree of freedom $(n = 1)$. The symmetry condition for the matrix $A_1(t) = (\sigma_1(t))$ is now satisfied automatically and condition 1° is reduced to the demand that $\sigma_1 = \lim_{t\to\infty}\sigma_1(t) \neq -2$ and negative. Condition 3° of regularity assumes, when $n = 1$, the form $\sigma_1^{\cdot 2} + |\sigma_1^{\cdot\cdot}| \in L_1(0, \infty)$. In the oscillatory case $0 > \sigma_1 > -2$, we shall obviously have $\nu_1 = \nu_2 = \sigma_1/2$ and condition 3° means that the estimate (2.5) must hold for any $m > 1$. In the aperiodic case $\sigma_1 < -2$

$$\nu_1 = (\sigma_1 + (\sigma_1^2 - 4)^{1/2})/2, \quad \nu_2 = (\sigma_1 - (\sigma_1^2 - 4)^{1/2})/2$$

and the number m in (2.5) must now satisfy the inequality

$$m > \frac{\sigma_1 - (\sigma_1^2 - 4)^{1/2}}{\sigma_1 + (\sigma_1^2 - 4)^{1/2}}$$

from which it follows that m must increase as σ_1 decreases in order to satisfy condition 3°.

4. To illustrate Theorem 1 we consider the case of a pendulum in the presence of viscous friction forces. The equation of motion has the form

$$q^{\cdot\cdot} + h(t)\, q^{\cdot} + \sin q = 0 \tag{4.1}$$

Equation (4.1) can be reduced to the set of equations (1.1) if we put

$$q^{\cdot} = p, \ H = 2^{-1}p^2 - \cos q + 1, \ Q(t, p, q) = -h(t)\,p$$

The matrix $A_1(t) = (-h(t))$ and conditions $1^\circ, 2^\circ$ reduce to the requirement that the following finite limit exists:

$$\lim_{t\to\infty} h(t) = h > 0, \ h \neq 2 \tag{4.2}$$

together with the inclusion

$$h^{\cdot 2} + |h^{\cdot\cdot}| \in L_1(0, \infty) \tag{4.3}$$

Since in the present case the non-linear part of the Lagrangian forces $Q_H(t, p, q) \equiv 0$, while $\Pi_1 = H - 2^{-1}(p^2 + q^2) = O(q^4)$ and $d\Pi_1/dq = O(|q|^3)$, it follows that the estimate (2.5) in condition 3° holds for $m = 3$. The characteristic roots $\lambda_{1,2}$ of the matrix

$$B = \lim_{t\to\infty} A(t) = \begin{Vmatrix} 0 & 1 \\ -1 & -h \end{Vmatrix}$$

have the form $\lambda_{1,2} = -2^{-1}h \pm (h^2/4 - 1)^{1/2}$, therefore condition 3° reduces to the demand that

$$0 < h < 4/\sqrt{3} \tag{4.4}$$

The values $h < 2$ correspond to the oscillatory case, and $-h > 2$ to the aperiodic case. Thus the conditions of applicability of Theorem 1 for Eq.(4.1) reduce to the condition that (4.2)–(4.4) hold.

Using Theorem 1 we shall write the expressions for the asymptotic forms of a two-parameter family of solutions for the oscillatory case $h<2$ (the aperiodic case is simpler and will not be considered). We construct the matrix $Y(t)$ using the matrix $Y_2(t)$ of (2.17) taking the real and imaginary part of the first column of $Y_2(t)$ as its first and second column. After simple reduction we obtain the following asymptotic representations for the solutions of (4.1) ($b_{1,2}$ are the coordinates of the vector c):

$$q(t)=\left(b_1\cos\int_{t_0}^{t}\eta_1(\tau)\,d\tau+b_2\sin\int_{t_0}^{t}\eta_1(\tau)\,d\tau+o(1)\right)\exp\int_{t_0}^{t}\nu_1(\tau)\,d\tau$$

$$p(t)=\left[(b_1\nu_1(t)+b_2\eta_1(t))\cos\int_{t_0}^{t}\eta_1(\tau)\,d\tau+(b_2\nu_1(t)-b_1\eta_1(t))\sin\int_{t_0}^{t}\eta_1(\tau)\,d\tau+o(1)\right]\exp\int_{t_0}^{t}\nu_1(\tau)\,d\tau$$

$$\nu_1(t)=\operatorname{Re}\lambda_1(t)=-h(t)/2,\ \eta_1(t)=\operatorname{Im}\lambda_1(t)=(1-h^2/4)^{1/2}$$

Note 3. The above method of construcing the real *FSM* $Y(t)$ of the linearized system (2.2) using the *FSM* of $Y_2(t)$, with the asymptotic form given by (2.17), can also be used in the general case of n degrees of freedom. If the roots $\lambda_{i,n+i}$ are not real (when $\lambda_i=\overline{\lambda}_{n+i}$), then we take the real and imaginary part of the i-th column of $Y_2(t)$ as the i-th and $(n+i)$-th column of $Y(t)$. On the other hand, if $\lambda_{i,n+i}$ are real, then we take as the i-th and $(n+i)$-th column of $Y(t)$ the corresponding columns of the matrix $1/2\,(Y_2(t)+\overline{Y}_2(t))$. The fact that this yields a real *FSM* of $Y(t)$ of the linearized system (2.2) can be confirmed using the asymptotic representation (2.17) for $Y_2(t)$.

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ON TWO TYPES OF SWIRLING GAS FLOWS*

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Two classes of exact solutions of the three-dimensional stationary equations of gas dynamics are constructed. The solutions are used to describe isentropic gas flows with two types of swirling in axisymmetric divergent channels. The effect of swirling on the thrust of special-type nozzles is studied.

Approximate analytic or numerical methods were used earlier to study radial-equilibrium flows with arbitrary swirling in /1/, and various qualitative features of the swirling flows, such as the appearance of vacuum kernels, back flows and stagnation zones at the inlet to the nozzle throat were discussed in /2–4/. Analytic solutions in the transonic approximation were constructed in /5/ and the dependence of the nozzle thrust on the swirling parameters were investigated in /1, 6–9/.

1. In studying swirling gas flows we will use two classes of solutions of the equations of gas dynamics for the case when the velocity vector components u_i and the function $Q=\rho^{\gamma-1}$ (ρ is density and γ is the adiabatic index) depend linearly on some spatial coordinates x_k /10/.

First we consider isentropic three-dimensional flows when the linear dependence on x_2 and x_3 has the form

$$Q=g(x_1),\quad u_1=g_1(x_1)\qquad(1.1)$$
$$u_i=l_i(x_1)x_2+f_i(x_1)x_3+g_i(x_1),\quad i=2,3$$

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